Analysing Algorithms (Part 2 - Examining Algorithms)

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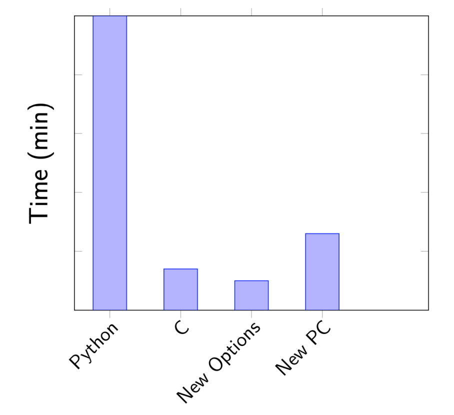
Lecture 08.1.2 (v1.0.2)

Signposting

- Analysing Algorithms is split into three parts:
  - Part 1: Motivation and Algorithmic Complexity
  - Part 2: Examining algorithms
  - Part 3: Turing Machines and Complexity Classes
- This is Part 2
- Thanks to Turing Fellow and Computer Scientist Dan Martin for Tikz pictures and expertise

Runtime vs Complexity - motivation



- Consider our algorithm run on data D_1:
  - In different programming languages, compile arguments and hardware
- What can be said in general?

Algorithm Example (1)

- What is the complexity of the following algorithm?

procedure $\textsc{Example}(a, b, n)$
  $i \leftarrow 1$
  while $i \leq n$ **do**
    $a \leftarrow f_1(b, n)$
    $b \leftarrow f_2(a, n)$
    $i \leftarrow i + 1$
  end while
  return b
end procedure

- $f_i(a, n)$ has runtime $T_i(n)$
- Inside loop is $\mathcal{O}(T_1(n) + T_2(n))$
- Total $\mathcal{O}[n(T_1(n) + T_2(n))]$

Algorithm Example (2)

- Compare to the following algorithm?

```
procedure EXAMPLE(a, b, n)
    i ← 1
    while i ≤ n do
        a ← f_1(b, n)
        b ← f_2(a, n)
        i ← 2 · i
    end while
    return b
end procedure
```

- Inside loop is $\mathcal{O}(T_1(n) + T_2(n))$
- Total $\mathcal{O}[\log(n)(T_1(n) + T_2(n))]$

Sorting examples

- We have some data: $1, 4, 6, 2, 3, 7, 5, \cdots$
- We want to sort the data into ascending order: $1, 2, 3, 4, 5, 6, 7, \cdots$
- What is the best[1] algorithm?
  - **Insertion sort** is $\Theta(n^2)$, but operates in-place.
  - **Merge sort** is $\Theta(n \log(n))$, but memory requirements grow with data size.
  - **Heap sort** is $\Theta(n \log(n))$ and sorts in place.
  - **Quick sort** is $\Theta(n^2)$, but $\Theta(n \log(n))$ expected time, and is often fastest in practice.
  - **Counting sort** allows array indices to be sorted in $\Theta(n)$ by exploiting knowledge that all integers are present.
  - **Bucket sort** is $\Theta(n^2)$, though $\Theta(n)$ average case (if data are Uniform!)

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[1] Cormen et al 2010 Introduction to Algorithms

Quicksort: a Recursion Example

```
procedure QuickSort(A)
    if len(A) == 1 then
        return A
    else
        x ← A
        A_l ← {a ∈ A : a < x}
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What if we can choose the **median element** of A?

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$T(n)$

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$$T(n) = 2T(\frac{n}{2}) + n$$

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What if we can choose the **median element** of A?

$$
\begin{aligned}
T(n) & \\
& = 2T(\frac{n}{2}) + n \\
& = 2(2T(\frac{n}{4}) + \frac{n}{2}) + n
\end{aligned}
$$

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What if we can choose the **median element** of A?

$T(n)$

$$\begin{aligned} &= 2T(\frac{n}{2}) + n \\ &= 2(2T(\frac{n}{4}) + \frac{n}{2}) + n \\ &= \dots \end{aligned}$$

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&= \Theta(n \log n)
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What if we always choose the **largest element** of A?

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$$T(n) = T(n-1) + n$$

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$$\begin{aligned} &= T(n-1)+n \\ &= (T(n-2)+n)+n \end{aligned}$$

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$$T(n)$$
$$\begin{aligned} &= T(n-1) + n \\ &= (T(n-2) + n) + n \\ &= \dots \\ &= T(1) + \sum_{i=1}^{n} i \end{aligned}$$

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\begin{aligned}
T(n) & \\
&= T(n-1) + n \\
&= (T(n-2) + n) + n \\
&= \dots \\
&= T(1) + \sum_{i=1}^{n} i \\
&= n(n-1)/2 = \Theta(n^2)
\end{aligned}
$$

Other types of complexity

- Complexity questions are primarily asked about:
  - **Computation** (time)
  - **Space** (memory)
  - **Communication** (data transfer)
- They are all studied analogously - it is the unit of counting that changes
- Despite that, the theory is quite different

Space complexity

- Simply the amount of memory that an algorithm needs
- You can calculate it simply by adding the memory allocations
- Space required is **additional** to the input, which is **not** counted - this can conceptually not be stored at all, as in e.g. streaming algorithms
- Formally defined in terms of the Turing Machine (8.1.3)
- It can often be traded for time complexity, e.g. by storing intermediate results vs revisiting the calculation
- For a Data Scientist, this trade off is critical!
- We use the same notation

Space complexity example (1)

- **Problem**: Find x, y in X s.t. $x + y = T$ (known to exist)
- Solution 1:

```
import heapq
heapq.heapsort(X)
i=0;j=n-1;
while(X[i]+X[j]!=T):
  if X[i]+X[j]<T:
      i=i+1
  else:
      j=j-1
```

- Heapsort has $\mathcal{O}(1)$ space complexity
- Therefore the whole algorithm is $\mathcal{O}(1)$ in space
- And time complexity $\mathcal{O}(n \log(n) + n) = \mathcal{O}(n \log(n))$

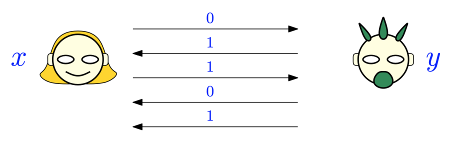
Space complexity example (2)

- Find x, y in X s.t. $x + y = T$ (known to exist)
- Solution 2:

```
D={}
for i in range(len(X)):
    D[T-X[i]]=i
for x in X:
    y=T-x
    if y in D:
        return X[D[y]],x
```

- This is $\mathcal{O}(n)$ in space
- Hash lookups are $\mathcal{O}(1)$ **average case** complexity ($\mathcal{O}(n)$ worst case - which does not apply here!)
- So this algorithm is $\mathcal{O}(n)$ in time too

Communication Complexity



- Alice knows $x \in X$, Bob knows $y \in Y$
- Together they want to compute $f(x, y)$ where $f \in X \times Y \to Z$
- Via a pre-arranged **protocol** P determining what they send
- The **communication cost** is the number of bits sent [2]

[2] Raznorov 2015 Communication Complexity Lecture

Communication Complexity

- The **Overall cost** of P is $\mathrm{C}(P) = \max_{x,y}[P(x,y)]$, i.e. the maximum possible cost for all data
- The **Communication complexity** of f is $\mathrm{C}(f) = \min_{P \in \mathcal{P}}(C[P(x,y)])$
- It is the minimum number of bits needed to compute $f(x,y)$ for any x, y
- Communication Complexity Theory describes $\mathrm{C}(f)$, typically by finding **bounds** (upper and lower) for a given f
  - Again typically as a function of the size of x and y, and always for some well defined spaces X and Y.
- Note that there is a trivial bound of $n+1$ for transferring all the data! (and then the answer back)

Communication Complexity Examples

- $f(x, y) = \text{Parity}([x, y])$
  - Parity$=mod_2(\sum_{i=1}^{n} x_i)$
  - $\text{C}(f(x, y)) = 2$ because Alice calculates the Parity of x, Bob the Parity of y, and they each communicate their own parity
- $f(x, y) = \text{Equality}(x, y)$
  - i.e. 1 if $x_i = y_i \quad \forall i$, and 0 otherwise
  - $\text{C}(f(x, y)) = n$ because every bit must be compared
- Typically approximate algorithms allow dramatically lower complexity
  - All the interesting theory is in this space

What is communication complexity theory good for?

- There are lots of immediate applications
  - Optimisation of computer networks
  - **Parallel algorithms**: communication between multiple cores on a CPU, or nodes of a cluster
  - And basically anything involving the internet!
  - Especially differential privacy (Block 12)
- There are many more less immediate applications
  - Particularly as a tool for algorithm and data structure lower bounds

Reflection

- What are the main subjects of complexity theory, and in which ways are they similar?
- By the end of the course, students should be able to:
  - Define three subjects of complexity theory
  - Apply each to simple algorithms, including compound algorithms
  - Reason about their value at a high level

Signposting

- Next up: Part 3: Turing Machines
- **References**:
  - Cormen et al 2010 Introduction to Algorithms
  - Toniann Pitassi Lecture on Communication Complexity: Applications and New Directions
  - Raznorov 2015 Communication Complexity Lecture
  - Arora and Barak Computational Complexity: A Modern Approach
    - One of few places to give space complexity much time (its always the poor cousin)