Analysing Algorithms (Part 1 - Complexity notation)

Daniel Lawson — University of Bristol

Lecture 08.1.1 (v1.0.3)

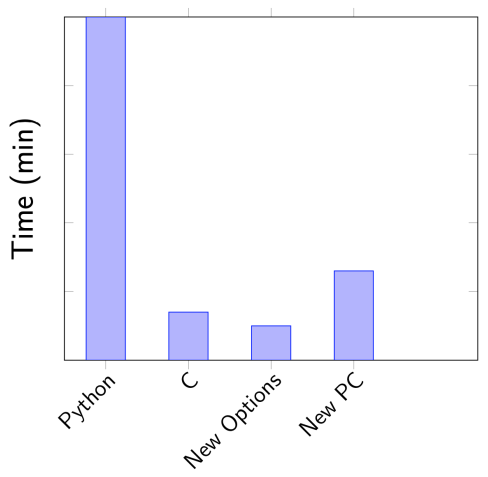
Signposting

- This set of lectures is about the conceptual framework for algorithms.
- Analysing Algorithms is split into three parts:
    - Part 1: Motivation and Algorithmic Complexity
    - Part 2: Examining algorithms
    - Part 3: Turing Machines and Complexity Classes
- This is Part 1
- We examine important algorithmic building blocks in 8.2.
- Thanks to Turing Fellow and Computer Scientist Dan Martin for Tikz pictures and expertise

ILOs

- ILO2 Be able to **use and apply basic machine learning** tools
- ILO4 Be able to use high throughput computing infrastructure and understand appropriate algorithms

Runtime - motivation



- Consider our algorithm run on data D_1:
- Different programming languages/compiler/hardware
- How do we predict its runtime elsewhere?

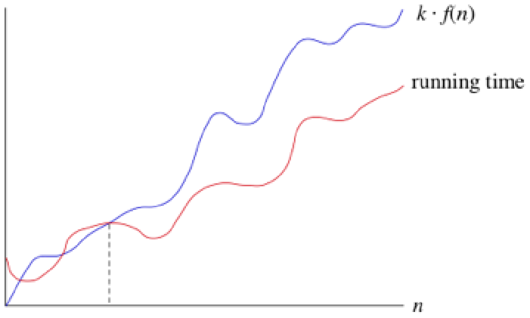
Why study algorithms?

- Algorithms underlie every machine-learning method.
- Theoretical statements about algorithms can be made, including:
    - How long does an algorithm take to run?
    - What guarantees can be made about the answer an algorithm returns?
- In some cases, **carefully chosen algorithms** can achieve either perfect or usefully good performance at a vanishing fraction of the run time of a naive implementation.
- This can lead to a **solution on a single machine** that is superior to that of a massively parallel implementation using distributed computing.

Algorithmic concerns

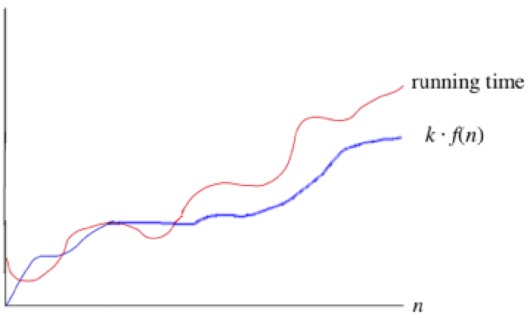
- We typically care about:
  - How long does the algorithm run for? Under which circumstances?
  - How do they trade off **runtime** and **memory requirement**?
- Some special values include **in-place** methods (which have a constant memory requirement) and **streaming** methods which visit the data exactly once each (usually with a constant-sized memory).
- Proofs typically describe the scaling of these properties, but in practice the constants are very important!

Algorithmic complexity: Big O Notation



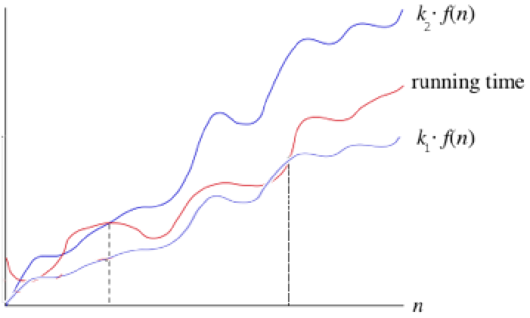
- $\mathcal{O}(n)$: An upper bound as a function of data size n
- $g(n) = \mathcal{O}(f(n))$:
  - $\exists n_0, k \in \mathbb{N}$ such that:
  - $\forall n \geq n_0$:
  - $g(n) \leq k \cdot f(n)$

Algorithmic complexity: Big Omega Notation



- $\Omega(n)$: A lower bound a function of data size n
- $g(n) = \Omega(f(n))$:
  - $\exists n_0, k \in \mathbb{N}$ such that:
  - $\forall n \geq n_0$:
  - $g(n) \geq k \cdot f(n)$

Algorithmic complexity: Big Theta Notation



- $\Theta(n)$: A tight bound as a function of data size n
- $g(n) = \Theta(f(n))$:
  - $\exists n_0, k_1, k_2 \in \mathbb{N}$ such that:
  - $\forall n \geq n_0$:
  - $k_1 \cdot f(n) \leq g(n) \leq k_2 \cdot f(n)$
- i.e. the bound is strict.

Complexity examples

- $n \in \mathcal{O}(n^2)$
  - $n \in \mathcal{O}(n)$ as well
  - $n \in \Omega(n)$
- $2n^2 + n + 10 \in \mathcal{O}(n^2)$
- $\log(n) \in \mathcal{O}(n^\epsilon)$ for all $\epsilon > 0$
- If $f(n) \in \mathcal{O}(g(n))$ then $g(n) \in \Omega(f(n))$
- If $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$ then $f(n) \in \Theta(g(n))$
- If $f_1(n) \in \mathcal{O}(g_1(n))$ and $f_2(n) \in \mathcal{O}(g_2(n))$ then $f_1(n) \cdot f_2(n) \in \mathcal{O}(g_1(n) \cdot g_2(n))$
- If $f_1(n) \in \mathcal{O}(g_1(n))$ and $f_2(n) \in \mathcal{O}(g_2(n))$ then $f_1(n) + f_2(n) \in \mathcal{O}(max(g_1(n), g_2(n)))$
- $2n^2 + 3n + 1 = 2n^2 + \Theta(n) = \Theta(n^2)$

Algorithmic complexity: Probabilistic Analysis

- Sometimes we don't want the worst-case behaviour out of all possible inputs
- In these scenarios **average-case** run time is often reported
  - This is typically the average over the entire input space
  - This should make the statistician in you concerned!
- Randomized algorithms are also important
  - In these the answer may be random, and take a random amount of time, for a given input!
  - e.g. MCMC, etc
  - Again the **expected run time** is often reported
- We can discuss Θ, Ω and $\mathcal{O}$ of the expected runtime
- Clearly the distribution of the input data is important
- Some worst-case scenarios have "measure 0" (i.e. will never occur in practice)

Complexity and constants

- Consider the following functions:

```
import time
def constant_fun(n,k):
    time.sleep(k * k);
def linear_fun(n,k):
    for i in range(n):
        time.sleep(1);
```

- Clearly `linear_fun` is faster for $n < k^2$. We need to take into account k and whether it scales with n.
- In practice k is often truly a constant but can be any scale compared to n. The accounting therefore needs to retain it.
- Example: SVD is $\mathcal{O}(\min(mn^2, m^2n))$
- Complexity classes **only** describe **asymptotic behaviour** for large n

Divide and conquer

- One of the most popular strategies is Divide and Conquer, in which we make many sub-problems, each of which is solvable.
- This is typically valuable for parallellism
- It also makes sense to apply the algorithm **recursively**.
    - In which case we obtain expressions like:

$$T(n) = aT(n/k) + D(n) \quad \text{if} \quad n \geq c,$$

    - and $T(n) = \Theta(1)$ otherwise.
    - This recursion is a relatively straightforward infinite sum (exercises) and leads to $T(n) = \Theta(n \log_k(n))$

Other key concepts

- **Worst case cost** conditions: can require care when looking up the answer.
  - For example, some data structures have $\mathcal{O}(n)$ lookup cost if the data are missing, but much better if the data are present.
  - Also some costs are predictable and rare, leading to…
- **Amortised cost**: The long term, average worst case cost, which is often better than the single case cost.
  - For example, some data structures must be periodically rebuilt when they get too big, an expensive action. But this is done rarely by construction.

Reflection

- Does it make sense to say that "$\mathcal{O}(f(n))$ is at least n^2"?
- In what sense would it matter in a recursive binary algorithm if n was not in 2^k?
- How do complexity statements combine?
- By the end of the course, you should:
    - Be able to compute with Θ, Ω and $\mathcal{O}$
    - Be able to reason at a high level about algorithm value

Signposting

- Next up: Analysing Algorithms Part 2: Examining algorithms
- **References**:
    - Wikipedia Divide and Conquer
    - Cormen et al 2010 Introduction to Algorithms is very accessible and recommended.
    - Arora and Barak 2007 Computational Complexity: A Modern Approach is useful but more advanced.